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High distance Heegaard splittings from involutions

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ABSTRACT

Fixed an oriented handlebody $H = H_+$ with boundary F, let $\eta(H_+) = H_-$ be the mirror image of H_+ along F, so $\eta(F)$ is the boundary of H_-, for a map $f : F \to F$, we have a 3-manifold by gluing H_+ and H_- along F with attaching map f, and denote it by $M_f = H_+ \cup_{f:F\to F} H_-$. In this note, we show that there are involutions $f : F \to F$ which are also reducible, such that M_f have arbitrarily high Heegaard distances.

1. Introduction and preliminaries

A Heegaard splitting of a closed orientable 3-manifold M is a decomposition of it along an embedded closed surface S into two handlebodies H_+ and H_-, and as a conscious extension of A. Casson and C. Gordon's notion of strong irreducibility, J. Hempel (see [5]) defined the Heegaard distance of a Heegaard splitting in terms of the curve complex $\mathscr{C}(S)$ of S, where the Heegaard distance is the minimal distance between two curves α_+ and α_- in $\mathscr{C}(S)$ which bound disks in H_+ and H_- respectively. By Perelman's proof of geometrization conjecture of Thurston, if a manifold admits a distance at least 3 Heegaard splitting, then the manifold is hyperbolic.

For any fixed genus g, there are arbitrarily high distance Heegaard splittings of genus g, see the construction given by Hempel [5] using generic pseudo-Anosov maps, J. Hempel wrote that the construction was inspired by the arguments of F. Luo, and F. Luo attributed some of his ideas to the work of T. Kobayashi [6].

1.1. The question

W. Thurston [10,3] classified the mapping classes of a surface by dynamics properties: reducible, periodic and pseudo-Anosov.

Fixed an oriented handlebody $H = H_+$ with boundary F, we assume that the orientation of H pointing out from F. Let $\eta(H_+) = H_-$ be the mirror image of H_+ along F, so $\eta(F)$ is the boundary of H_-, and the orientation of H_- pointing in from $\eta(F)$, a curve c in F bounds a disk in H_+ if and only if $\eta(c)$ bounds a disk in H_-. For a map $f : F \to F$, we have a manifold by gluing H_+ and H_- along F with the map $\eta \circ f$, we abuse the notion and denote it by $M_f = H_+ \cup_{f:F\to F} H_-$, an example is that $M_{id} = \sharp^g S^2 \times S^1$.

In [2], J. Birman asked the following question:

Problem. How is the Nielsen–Thurston trichotomy related to the question of whether the Heegaard distance is 0, 1, 2 or 3?

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It is proved by A. Abrams and S. Schleimer [1] that for a generic pseudo-Anosov f, the Heegaard distances of M_{f^n} increase roughly linearly with n.

In this short note, we show that there are involutions $f : F \to F$ which are also reducible, such that M_f have arbitrarily high Heegaard distances, so the relationship between Nielsen–Thurston trichotomy and Hempel's distance is subtle.

1.2. Masur–Minsky theory

Let F be a compact surface with $\chi(F) \leqslant -2$, Harvey [4] defined the *curve complex* $\mathscr{C}(F)$ as follows: The vertices of $\mathscr{C}(F)$ are the isotopy classes of essential simple closed curves on F, and $k+1$ distinct vertices $c_0, c_1, \ldots, c_k$ determine a k-simplex of $\mathscr{C}(F)$ if and only if they are represented by pairwise disjoint simple closed curves. $\mathscr{C}(F)$ can be made into a complete geodesic metric space in a natural way by making each simplex a regular Euclidean simplex of side-length 1, and for two vertices a and b of $\mathscr{C}(F)$, the distance of a and b, denoted by $d_{\mathscr{C}(F)}(a, b)$, is well defined. For two sets of vertices in $\mathscr{C}(F)$, $d_{\mathscr{C}(F)}(A, B)$ is defined to be $min\{d_{\mathscr{C}(F)}(a, b) \mid a \in A,\ b \in B\}$. Now let F be a once-punctured torus (or once-holed torus), in this case, Masur and Minsky [7] modified the definition of $\mathscr{C}(F)$ as follows: the vertices of $\mathscr{C}(F)$ are the isotopy classes of essential simple closed curves on F, and $k+1$ distinct vertices $c_0, c_1, \ldots, c_k$ determine a k-simplex of $\mathscr{C}(F)$ if and only if c_i and c_j are represented by two simple closed curves x_i and x_j on F such that x_i intersects x_j in just one point for pairs (i, j) with $0 \leqslant i, j \leqslant k$. In this case, $\mathscr{C}(F)$ is two-dimensional, and $\mathscr{C}(F)^1$ is isometric to the well-known Farey graph.

A celebrated theorem of Masur and Minsky [7] is the following:

Theorem 1.1. *$\mathscr{C}(F)$ is δ-hyperbolic in the sense of Gromov and Cannon, where δ depends only on the topology of F.*

For the handlebody H, we denote its boundary by F, and we denote by $\mathscr{D} = \mathscr{D}(H)$ the subset of $\mathscr{C}(F)^0$ such that each element of $\mathscr{D}(H)$ bounds a disk in H, we also call $\mathscr{D}$ the *disk set* of H.

Definition 1.2. See [5], for a Heegaard splitting $M = W \cup_S V$, the *Heegaard distance* is $d_{\mathscr{C}(S)}(\mathscr{D}(W), \mathscr{D}(V))$.

Let A be a subset of a metric space X, A is said to be *κ-quasi-convex* if there is a constant κ, such that $\forall a, b \in A$, any geodesic $[a, b]$ is in the κ-neighborhood of A.

The following theorem of [9] is important for our arguments:

Theorem 1.3. *Let H be a genus g handlebody, then there is a constant κ which is determined by g, such that the disk set $\mathscr{D}(H)$ is κ-quasi-convex in $\mathscr{C}(\partial H)$.*

Let F be a compact surface of genus at least one, and Y be a compact subsurface of F such that $g(Y) \geqslant 1$, we say Y is essential on F if the induced map of the inclusion from $\pi_1(Y)$ to $\pi_1(F)$ is injective. See [8] for the detailed definition of essential subsurface.

We denote by $\mathscr{P}(\mathscr{C}(Y))$ the power set of $\mathscr{C}(Y)$. Let $c \in \mathscr{C}(F)^0$, we denote by x the essential simple closed curve on F which represents c. Masur and Minsky defined the *subsurface projection map* π_Y from $\mathscr{C}(F)$ to $\mathscr{P}(\mathscr{C}(Y))$ as follows: If $x \cap Y = \emptyset$, then $\pi_Y(c) = \emptyset$; if $x \subset Y$, then $\pi_Y(c) = c$; suppose now that $x \cap Y \neq \emptyset$ and x does not lie in Y, then $x \cap Y$ contains k arc components in Y, say $x_1, \ldots, x_k$. We denote by $\eta(x_i)$ the open regular neighborhood of x_i in Y. Let $Y^i = Y - \eta(x_i)$. We denote by d_i the set of vertices of $\mathscr{C}(Y)$ represented by $\partial Y^i - \partial Y$, note that each d_i consists of at most two disjoint curves, let $\pi_Y(c) = \bigcup_{i=1}^k d_i$, which is the subsurface projection map from $\mathscr{C}(F)$ to $\mathscr{C}(Y)$. It is easy to see that the set $\bigcup_{i=1}^k d_i$ has diameter at most 2 in $\mathscr{C}(Y)$, actually, the following is true [8]:

Lemma 1.4. *The subsurface projection map is 2-Lipschitz.*

Another key theorem we must use is Masur–Minsky's Bounded Geodesic Image Theorem [8]:

Theorem 1.5. *Let Y be an essential subsurface of F, and let γ be a geodesic segment, ray, or bi-infinite line in $\mathscr{C}(F)$, such that $\pi_Y(v) \neq \emptyset$ for every vertex v of γ. There is a constant $\mathfrak{M}$ depending only on F so that $diam_{\mathscr{C}(Y)}(\pi_Y(\gamma)) \leqslant \mathfrak{M}$.*

2. The proof of the theorem

Theorem 2.1. *For any large $n \in \mathbb{N}$, there is $f : F \to F$, which is a reducible involution, such that $M_f = H_+ \cup_f H_-$ has Heegaard distance at least n.*

Proof. We first assume that $g(F) = 2k$ is even, then there is separating curve c in F which cuts F into two surfaces F_1 and F_2, and $g(F_1) = k = g(F_2)$. Since there are Heegaard splittings of genus g with arbitrarily high Heegaard distances [5], we can choose c such that $d(c, \mathscr{D}) > n > \kappa + 2$, where κ is the quasi-convexity constant in Theorem 1.3. Fixed an orientation preserving homeomorphism $\tau : F_1 \to F_2$ with $\tau|_c = id$, and given $f_1 : F_1 \to F_1$ an orientation preserving pseudo-Anosov map with $f_1|_c = id$, let $f : F \to F$ be the orientation preserving map such that $f|_{F_1} = \tau \circ f_1 : F_1 \to F_2$, and $f|_{F_2} = f_1^{-1} \circ \tau^{-1} : F_2 \to F_1$, then $f^2 = id$.

Take any pair $a, b \in \mathscr{D}$, let g be a geodesic in $\mathscr{C}(F)$ connecting a to b, and x be any vertex of g. We claim that $x \cap F_i \neq \emptyset$ for $i = 1, 2$. Suppose otherwise, that $x \cap F_1 = \emptyset$, then $d_{\mathscr{C}(F)}(x, c) \leqslant 1$, and by Theorem 1.3, there is an $x' \in \mathscr{D}$ such that $d_{\mathscr{C}(F)}(x, x') \leqslant \kappa$, so we have $d(\mathscr{D}, c) \leqslant \kappa + 1$, a contradiction. Then by the Bounded Geodesic Image Theorem, for this geodesic g, we have $d_{\mathscr{C}(F_i)}(\pi_{F_i}(a), \pi_{F_i}(b))$ is bounded above uniformly by the constant $\mathfrak{M}$, so $\pi_{F_i}(\mathscr{D}) \subset \mathscr{C}(F_i)$ has bounded image by the constant $\mathfrak{M}$. And then, take a high power of f_1, we abuse notion and denote it also by f_1, using the North–South dynamics of pseudo-Anosov maps, we can assume that $d_{\mathscr{C}(F_1)}(f_1^{-1}\tau^{-1}(\pi_{F_2}(\mathscr{D})), \pi_{F_1}(\mathscr{D})) > 2n$.

We now prove that $M_f = H_+ \cup_f H_-$ has Heegaard distance at least n. Note that $f(\mathscr{D})$ is the set of curves which bounds disk in H_-, suppose that $a, b \in \mathscr{D}$ such that a and $f(b)$ realize the Heegaard distance of $M_f = H_+ \cup_f H_-$, and g is a geodesic in $\mathscr{C}(F)$ connecting a to $f(b)$. Arguments by contradiction, we assume $d(a, f(b)) < n$, then for any x which is a vertex of g, $x \cap F_1 \neq \emptyset$ and $x \cap F_2 \neq \emptyset$, otherwise, we have $d_{\mathscr{C}(F)}(a, c) \leqslant d_{\mathscr{C}(F)}(a, x) + 1 \leqslant n$, a contradiction. Then by Lemma 1.4, we have $d_{\mathscr{C}(F_1)}(\pi_{F_1}(a), \pi_{F_1}(f(b))) \leqslant 2n$, note that $f_1^{-1}\tau^{-1}(\pi_{F_2}(b)) = \pi_{F_1}(f(b))$, so a contradiction to $d_{\mathscr{C}(F_1)}(f_1^{-1}\tau^{-1}(\pi_{F_2}(\mathscr{D})), \pi_{F_1}(\mathscr{D})) > 2n$.

For the $g(F) = 2k + 1$ case, we take two non-separating curves c_1 and c_2 which decompose F into two surfaces F_1 and F_2 with genus k, then the above arguments follow. $\square$

Remark 2.2. For any $d \in \mathbb{N}$, there are some large numbers g, such that for the genus g handleboby H, there are period d maps $f : F \to F$, such that M_f have high arbitrarily high Heegaard distances: for example, we can take $g = nd$, and we take $c_1, c_2, \ldots, c_d \subset F$ to be a set of pairwise disjoint separating curves, such that $F - \bigcup_{i=1}^{d} c_i$ is compose of a genus zero surface F_0 with d boundary $c_1, c_2, \ldots, c_d$, and genus n surfaces F_i with one boundary c_i for $i = 1, 2, \ldots, d$. Then fixed homeomorphisms $\tau_i : F_i \to F_{i+1}$ mod d for $i \geqslant 1$ such that $\tau_d \cdots \tau_2 \tau_1 = id : F_1 \to F_1$, a periodic map $f_0 : F_0 \to F_0$ with $f_0(c_i) = c_{i+1}$ mod d and a pseudo-Anosov map $f_1 : F_1 \to F_1$, using τ_i, f_0 and f_1 we can make a periodic map $f : F \to F$ such that $f(F_0) = F_0$ and $f|_{F_i} : F_i \to F_{i+1}$ mod d, i.e., $f|_{F_1} = \tau_1 f_1 : F_1 \to F_2$, $f|_{F_2} = \tau_2 \tau_1 f_1 \tau_1^{-1} : F_2 \to F_3$, $f|_{F_3} = \tau_3 \tau_2 \tau_1 f_1 \tau_1^{-1} \tau_2^{-1} : F_3 \to F_4, \ldots$, but for $f|_{F_d}$, we have $f|_{F_d} = \tau_d \cdots \tau_2 \tau_1 f_1^{-(d-1)} \tau_1^{-1} \tau_2^{-1} \cdots \tau_{d-1}^{-1} : F_d \to F_1$, then it can be showed that $f^d = id$. We can take c_i which have large distances with $\mathscr{D}$ and a sufficiently high power of f_1 as in the proof of Theorem 1.2 such that our claim can follow.

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